Journal of Mathematical Sciences: Advances and Applications
Volume 3, Number 1, 2009, Pages 121-128

SOLUTIONS FOR A CLASS OF FIFTH-ORDER NONLINEAR PARTIAL DIFFERENTIAL SYSTEM

CESAR A GOMEZ S[1] and ALVARO SALAS[2]

[1]Department of Mathematics
Universidad Nacional de Colombia
Sede Bogotá
Colombia
e-mail: cagomezsi@unal.edu.co

[2]Department of Mathematics
Universidad de Caldas
Universidad Nacional de Colombia
Manizales, Colombia
e-mail: asalash@unal.edu.co

Abstract

In this paper, we use the generalized tanh method to obtain exact solutions for a class of fifth-order nonlinear systems. A particular case is given by the integrable Mikhailov-Novikov-Wang system (MNW). Periodic and solitons solutions are formally derived. The Mathematica is used.

1. Introduction

The class of fifth-order systems that we consider reads

$$\begin{cases} u_t + \rho u_{xxxxx} + \gamma u u_{xxx} + \beta u_x u_{xx} + \alpha u^2 u_x - w_x = 0 \\ w_t + 6 w u_{xxx} + 2 u_{xx} w_x - 96 w u u_x - 16 w_x u^2 = 0, \end{cases} \tag{1}$$

2000 Mathematics Subject Classification: 35C05.

Keywords and phrases: PDE, nonlinear equation, fifth-order systems of PDE's, generalized tanh method, exact solutions, Mathematica.

Received February 3, 2009

where α, β, γ and ρ are arbitrary nonzero and real parameters, and $u = u(x, t)$, $w = w(x, t)$ are differentiable functions. Lots of forms of (1) can be constructed by changing the values of parameter. In the particular case $\alpha = -80, \beta = 50, \gamma = 20, \rho = -1$, (1) reduces to the new Mikhailov-Novikov-Wang system ([8, 9, 10, 13, 15]) (MVW).

$$\begin{cases} u_t = u_{xxxxx} - 20uu_{xxx} - 50u_x u_{xx} + 80u^2 u_x + w_x \\ w_t = -6wu_{xxx} - 2u_{xx}w_x + 96wuu_x + 16w_x u^2, \end{cases} \tag{2}$$

which was derived by the authors in [13] using the symmetry approach. We refer to [13] for more details about this system. In the case that $w(x, t) = 0$ and $\rho = 1$, the system (1) reduces to fifth-order KdV equation [1, 4, 6, 14, 16]

$$u_t + u_{xxxxx} + \gamma uu_{xxx} + \beta u_x u_{xx} + \alpha u^2 u_x = 0. \tag{3}$$

This equation has been recently studied in [4, 6, 14, 16]. Some important particular cases of (3) are given by Lax, Sawada-Kotera, Kaup-Kupershdmit, and Ito equations [4, 6, 14, 16].

The searching of exact solutions of nonlinear partial differential equations is of great importance for many researches. A variety of powerful methods such as tanh method and generalized tanh method [1, 2, 7, 16], general projective Riccati equations method [11, 12, 17] and other methods (see for instance [3, 4, 5, 6, 14, 16]) have been developed in this direction. In this work, we will use the generalized tanh method [2, 6, 7] to construct periodic and solitons solutions to (1). This paper is organized as follows: In Section 1, we will review briefly the generalized tanh method. In Section 2, we give the mathematical framework to search exact solutions to (1). In Section 3, we obtain exact solutions to (1). Finally some conclusions are given.

2. The Generalized Tanh Method

The generalized tanh method can be summarized as follows. For a given nonlinear equation that does not explicitly involve independent variables

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{4}$$

we use the wave transformation

$$u(x, t) = v(\xi), \quad \xi = x + \lambda t, \tag{5}$$

where λ is a constant.

Under the transformation (5), (4) reduces to ODE in the function $v(\xi)$

$$P(v, v', v'', \dots) = 0. \tag{6}$$

The next crucial step is to introduce a new variable $\phi(\xi)$ which is a solution of the Riccati equation

$$\phi'(\xi) = \phi(\xi)^2 + k, \tag{7}$$

where k is a constant.

It is well-known that the solutions of the equation (7) are given by

$$\phi(\xi) = \begin{cases} -\dfrac{1}{\xi}, & k = 0, \\ \sqrt{k}\tan(\sqrt{k}\xi) & k > 0, \\ -\sqrt{k}\cot(\sqrt{k}\xi) & k > 0, \\ -\sqrt{-k}\tanh(\sqrt{-k}\xi) & k < 0, \\ -\sqrt{-k}\coth(\sqrt{-k}\xi) & k < 0. \end{cases} \tag{8}$$

We seek solutions to (6) in the form

$$\sum_{i=0}^{M} a_i \phi(\xi)^i, \tag{9}$$

where $\phi(\xi)$ satisfies the Riccati Equation (7), and a_i are unknown constants. The integer M can be determined by balancing the highest derivative term with nonlinear terms in (6), before the a_i can be computed. Substituting (9) along with (7) into (6) and collecting all terms with the same power $\phi(\xi)^i$, we get a polynomial in the variable $\phi(\xi)$. Equaling the coefficients of this polynomial to zero, we can obtain a system of algebraic equations, from which the constants a_i, λ $(i = 1, 2, \ldots, M)$ are obtained explicitly. Lastly, we find solutions for (4) in the original variables.

3. Exact Solutions to (1)

To search exact solutions of the system (1), we use the traveling wave transformation

$$\begin{cases} u(x, t) = v(\xi), \\ w(x, t) = w(\xi), \\ \xi = x + \lambda t. \end{cases} \tag{10}$$

Under the transformation (10), (1) reduces to following nonlinear ordinary differential equations system with constant coefficients

$$\begin{cases} \lambda v'(\xi) + \rho v^{(5)}(\xi) + \gamma v(\xi) v^{(3)}(\xi) + \beta v'(\xi) v''(\xi) + \alpha v^2(\xi) v'(\xi) - w'(\xi) = 0, \\ \lambda w'(\xi) + 6w(\xi) v'''(\xi) + 2v''(\xi) w'(\xi) - 96 w(\xi) v(\xi) v'(\xi) - 16 w'(\xi) v^2(\xi) = 0. \end{cases} \tag{11}$$

From the first equation in (11) we obtain

$$w'(\xi) = \lambda v'(\xi) + \rho v^{(5)}(\xi) + \gamma v(\xi) v^{(3)}(\xi) + \beta v'(\xi) v''(\xi) + \gamma v^2(\xi) v'(\xi), \tag{12}$$

which can be written as

$$\left(\lambda v(\xi) + \frac{\alpha}{3} v^3(\xi) + \frac{\beta - \gamma}{2} (v'(\xi))^2 + \gamma v(\xi) v''(\xi) + \rho v^{(4)}(\xi) - w(\xi)\right)' = 0. \tag{13}$$

Integrating (13) once with respect to ξ we obtain

$$\lambda v(\xi) + \frac{\alpha}{3} v^3(\xi) + \frac{\beta - \gamma}{2} (v'(\xi))^2 + \gamma v(\xi) v''(\xi) + \rho v^{(4)}(\xi) - w(\xi) = c, \tag{14}$$

where c is integration constant. We take $c = 0$. Due to (14)

$$w(\xi) = \lambda v(\xi) + \frac{\alpha}{3} v^3(\xi) + \frac{\beta - \gamma}{2} (v'(\xi))^2 + \gamma v(\xi) v''(\xi) + \rho v^{(4)}(\xi). \tag{15}$$

Substituting (15) and (12) in the second equation in (11) and after simplifications we obtain the following ordinary differential equation

$$\begin{cases} \lambda^2 v'(\xi) + \lambda\rho v^{(5)} + \lambda(6+\gamma) v(\xi) v'''(\xi) + \lambda(\beta+2) v'(\xi) v''(\xi) + \lambda(\gamma - 112) v^2(\xi) v'(\xi) \\ + 2\rho v''(\xi) v^{(5)}(\xi) + 8\gamma v(\xi) v''(\xi) v'''(\xi) + 2\beta v'(\xi)(v''(\xi))^2 - 2(47\gamma + 8\beta) v^2(\xi) v'(\xi) v''(\xi) \\ - 16\rho v^2(\xi) v^{(5)} + 2(\alpha - 8\gamma) v^3(\xi) v'''(\xi) - 16(2\alpha + \gamma) v^4(\xi) v'(\xi) + 6\rho v'''(\xi) v^{(4)} \\ + 3(\beta - \gamma)(v'(\xi))^2 v'''(\xi) - 96\rho v(\xi) v'(\xi) v^{(4)} - 48(\beta - \gamma) v(\xi)(v'(\xi))^3 = 0. \end{cases} \tag{16}$$

According to the described above method, we seek solutions to (16) as

$$v(\xi) = a_0 + a_1 \phi(\xi) + a_2 (\phi(\xi))^2, \tag{17}$$

where $\phi(\xi)$ satisfies (7). Substituting (17) into (16) and using (7) we obtain a polynomial in $\phi(\xi)$. Equaling the coefficients of this polynomial to zero, and solving the resulting algebraic system respect to unknowns variables k, λ, a_0, a_1 and a_2 with aid the Mathematica we find the following sets of solutions:

- $a_1 = 0,\ a_2 = \frac{3}{4},\ k = \sqrt{\lambda},\ a_0 = \frac{\sqrt{\lambda}}{2}.$
- $a_1 = 0,\ a_2 = \frac{3}{4},\ k = -\sqrt{\lambda},\ a_0 = -\frac{\sqrt{\lambda}}{2}.$

Using (8) we obtain the following solutions to (16):

For $\lambda > 0$:

1. $v_1(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\cot^2(\lambda^{\frac{1}{4}}\xi).$
2. $v_2(\xi) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\coth^2(\lambda^{\frac{1}{4}}\xi).$
3. $v_3(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tan^2(\lambda^{\frac{1}{4}}\xi).$

4. $v_4(\xi) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tanh^2(\lambda^{\frac{1}{4}}\xi).$

Therefore by (10) and (15) the exact solutions to system (1) are given by :

For $\lambda > 0$:

1. $u_1(x, t) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\cot^2(\lambda^{\frac{1}{4}}(x + \lambda t)).$

$w_1(x, t) = \frac{\lambda^{\frac{3}{2}}}{192}(-48 - \alpha + 9\csc^2(\lambda^{\frac{1}{4}}(x + \lambda t)(\alpha + 16(1 + \gamma + 16\rho) +$

$3(\alpha + 8(\beta + 2\gamma + 80\rho))\cot^2(\lambda^{\frac{1}{4}}(x + \lambda t)\csc^2(\lambda^{\frac{1}{4}}(x + \lambda t)))))).$

2. $u_2(x, t) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\coth^2(\lambda^{\frac{1}{4}}(x + \lambda t)).$

$w_2(x, t) = \frac{\lambda^{\frac{3}{2}}}{192}(48 + \alpha + 9\operatorname{sech}^2(\lambda^{\frac{1}{4}}(x + \lambda t)(\alpha + 16(1 + \gamma + 16\rho) + 3(\alpha +$

$8(\beta + 2\gamma + 80\rho))\coth^2(\lambda^{\frac{1}{4}}(x + \lambda t)\operatorname{csch}^2(\lambda^{\frac{1}{4}}(x + \lambda t)))))).$

3. $u_3(x, t) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tan^2(\lambda^{\frac{1}{4}}(x + \lambda t)).$

$w_3(x, t) = \frac{\lambda^{\frac{3}{2}}}{192}(-48 - \alpha + 9\sec^2(\lambda^{\frac{1}{4}}(x + \lambda t)(\alpha + 16(1 + \gamma + 16\rho) + 3(\alpha +$

$8(\beta + 2\gamma + 80\rho))\sec^2(\lambda^{\frac{1}{4}}(x + \lambda t)\tan^2(\lambda^{\frac{1}{4}}(x + \lambda t)))))).$

4. $u_4(x, t) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tanh^2(\lambda^{\frac{1}{4}}(x + \lambda t)).$

$w_4(x, t) = \frac{\lambda^{\frac{3}{2}}}{192}(48 + \alpha - 9\operatorname{sech}^2(\lambda^{\frac{1}{4}}(x + \lambda t))(\alpha + 16(1 + \gamma + 16\rho)) + 27(\alpha +$

$8(\beta + 2\gamma + 80\rho))\operatorname{sech}^4(\lambda^{\frac{1}{4}}(x + \lambda t)) - 27(\alpha + 8(\beta + 2\gamma + 80\rho))\operatorname{sech}^6(\lambda^{\frac{1}{4}}(x + \lambda t))).$

4. Conclusions

In the previous sections, we have presented an analysis over a generalization of the MNW system. A large number of forms of the fifth-order system with exact solutions can be constructed. Exact solutions for some particular cases such that Mikhailov-Novikov-Wang system can be derived. The generalized tanh method provides a straightforward algorithm to compute particular periodic and solitons solutions for a large class of nonlinear systems.

References

[1] D. Baldwin, U. Goktas, W. Hereman, L. Hong, R. S. Martino and J. C. Miller, Symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDFs, J. Symbolic Compt. 37(6) (2004), 669-705.

[2] E. Fan and Y. C. Hon, Generalized tanh method extended to special types of nonlinear equations, Z. Naturforsch. A 57(8) (2002), 692-700.

[3] C. A. Gomez and A. Salas, The Cole Hopf transformation and improved tanh-coth method applied to new integrable system (KdV6), Appl. Math. Comp. 204(2) (2008), 957-962.

[4] C. A. Gomez and A. H. Salas, The generalized tanh-coth method to special types of the fifth-order KdV equation, Appl. Math. Comp. 203(2) (2008), 873-880.

[5] C. A. Gomez and A. Salas, Exact solutions to a new integrable system (KdV6), J. of Math. Sciences: Advances and Applications 1(2) (2008), 305-310.

[6] C. A. Gomez, Special forms of the fifth-order KdV equation with new periodic and soliton solutions, Appl. Math. Comp. 189(2) (2007), 1066-1077.

[7] C. A. Gomez, Exact solutions for a reaction diffusion equation by using the generalized tanh method, Scientia Et Technica. 35(8) (2007), 409-410.

[8] C. A. Gomez, A new travelling wave solution of the Mikhailov-Novikov-Wang system using the extended tanh method, Boletin de Matematicas 14(1) (2007), 38-43.

[9] C. A. Gomez, New exact solutions of the Mikhailov-Novikov-Wang system, International Journal of Computer, Mathematical Sciences and Applications 1 (2007), 137-143.

[10] C. A. Gomez, Exact solutions for a new fifth-order integrable system, Revista Colombiana de Matematicas 40 (2006), 119-125.

[11] C. A. Gomez and A. Salas, Exact solutions for the generalized shallow water wave equation by the general projective Riccati equations method, Boletin de Matematicas 13(1) (2006), 50-56.

[12] C. A. Gomez and A. Salas, New exact solutions for the combined sinh-cosh-Gordon equation, Lecturas Matematicas, special issue (2006), 87-93.

[13] A. V. Mikhailov, V. Novikov and J. P. Wang, on clasification of integrable non-evolutionary equation, Studies in Applied Mathematics 118(4) (2007), 419-457.

[14] A. H. Salas and C. A. Gomez, Computing exact solutions for some fifth KdV equations with forcing term, Appl. Math. Comp. 204(1) (2008), 257-260.

[15] A. Sergyeyev, Zero curvature representation for a new fifth-order integrable system, 12(7) (2006), 227-229.

[16] A. M. Wazwaz, The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, Appl. Math. Comp. 84(2) (2007), 1002-1014.

[17]Z. Yan, The Riccati equation with variable coefficients expansion algorithm to find more exact solutions of nonlinear differential equation, Comput. Phys. Comm. 152 (1) (2003), 1-8. Prepint version available at

http://www.mmrc.iss.ac.cn/pub/mm22.pdf/20.pdf

■